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Considerations concerning the contributions of fundamental particles to the vacuum energy density

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Abstract. The covariant regularization of the contributions of fundamental particles to the vacuum energy density is implemented in the Pauli–Villars, dimensional regularization, and Feynman regulator frameworks. Rules of correspondence between dimensional regularization and cutoff calculations are discussed. Invoking the scale invariance of free field theories in the massless limit, as well as consistency with the rules of correspondence, it is argued that quartic divergences are absent in the case of free fields, while it is shown that they arise when interactions are present.

1 Introduction

It has been pointed out by several authors that one of the most glaring contradictions in physics is the enormous mismatch between the observed value of the cosmological constant and estimates of the contributions of fundamental particles to the vacuum energy density [1]. Specifically, the observed vacuum energy density in the universe is approximately $0.73\rho_c$, where $\rho_c = 3H^2/8\pi G_N \approx 4\times10^{-47}\,\mathrm{GeV}^4$ is the critical density, while estimates of the contributions of fundamental particles range roughly from $(\mathrm{TeV})^4$ in broken supersymmetry scenarios to $(10^{19}\,\mathrm{GeV})^4 = 10^{76}\,\mathrm{GeV}^4$ if the cutoff is chosen to coincide with the Planck scale. Thus, there is a mismatch of roughly 59 to 123 orders of magnitude!

The aim of this paper is to discuss the nature of these contributions by means of elementary arguments.

In the case of free particles, it is easy to see that a covariant regularization is needed, which we implement in the Pauli–Villars (PV) [2, 3], dimensional regularization (DR) [4], and Feynman regulator (FR) [3] frameworks.

We recall that the vacuum energy density is given by

$$\rho = \langle 0|T_{00}|0\rangle\,, \tag{1}$$

where $T_{\mu\nu}$ is the energy-momentum tensor.

Defining $t_{\mu\nu} \equiv \langle 0|T_{\mu\nu}|0\rangle$ and assuming the validity of Lorentz invariance, we have

$$t_{\mu\nu} = \rho\; g_{\mu\nu}\,, \tag{2}$$

or, equivalently,

$$t_{\mu\nu} = \frac{g_{\mu\nu}}{4}\; t^{\lambda}{}_{\lambda}\,, \tag{3}$$

where we employ the metric

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1.$$

We first consider the case of a free scalar field. Expanding the fields in plane waves with coefficients expressed in terms of creation and annihilation operators, and using their commutation relations, one readily finds the familiar expression

$$\rho = t_{00} = \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\;\frac{\omega_k^2}{\omega_k}\,, \tag{4}$$

where $\omega_k = \left[(\mathbf{k})^2 + m^2\right]^{1/2}$, as well as

$$p = t_{ii} = \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\;\frac{(k_i)^2}{\omega_k}\,. \tag{5}$$

In (5), $i = 1, 2, 3$ and there is no summation over i.

Both (4) and (5) are highly divergent and therefore mathematically undefined. Moreover, as recently emphasized by Akhmedov [5], the usual procedure of introducing a three-dimensional cutoff leads to an obvious contradiction: since the integrands in (4) and (5) are positive, one would reach the conclusion that t_{00} and t_{ii} have the same sign, in contradiction with (2)! This reflects the fact that a three-dimensional cutoff breaks Lorentz invariance. Clearly, covariant regularization procedures are required!

This is an important issue, since expressions that are not properly regularized are often deceptive. A classical example is provided by the calculation of vacuum polarization in QED, in which a quadratically divergent contribution turns out to be zero upon the imposition of electromagnetic current conservation [6]. Similarly, the same requirement transforms linearly divergent contributions to the triangle diagrams into convergent ones [7]. In fact, it is important that the regularization procedure respects

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the symmetries and partial symmetries of the underlying theory.

The plan of this paper is the following. In Sect. 2, we discuss rules of correspondence between the position of the poles in DR and cutoff calculations. In Sect. 3 we implement the covariant regularization of $t_{\mu\nu}$ in the PV, DR, and FR frameworks, starting from (2), (4) and (5). In Sect. 4 we consider the evaluation of $t^{\lambda}{}_{\lambda}$ on the basis of well-known expressions for the vacuum expectation value of products of free field operators, as well as Feynman diagrams. Throughout the paper the role played by the scale invariance of free field theories in the massless limit is emphasized. Section 5 illustrates the important effects arising from interactions by means of two specific examples. Section 6 presents the conclusions. Appendix A proves a general theorem concerning the signs of the vacuum energy density contributions of a free scalar field when it is regularized in the PV framework with the minimum number of regulator fields, while Appendix B illustrates the rules of correspondence in the evaluation of the one-loop effective potential. Section 4 contains a phenomenological update of the Veltman–Nambu sum rule for m_H [8, 9] and of an alternative relation discussed in [10].

2 Rules of correspondence

Since DR does not involve cutoffs explicitly, this approach is seldom employed in discussions concerning the cosmological constant and hierarchy problems. However, as it will be shown, it does give valuable information about the nature of the ultraviolet singularities. Furthermore, it has other important virtues for the problems under consideration: it respects the scale invariance of free field theories in the massless limit, does not involve unphysical regulator fields, and it is relatively easy to use in the two-loop calculation carried out in Sect. 5.

In order to discuss the position of the poles corresponding to specific ultraviolet divergences in multi-loop calculations, it is convenient to multiply each n-dimensional integration $\int d^n k$ by μ^{4-n}, where μ is the 't Hooft scale. This ensures that the combination of the prefactor and the integration has the canonical dimension 4.

Let us first consider quadratic ultraviolet divergences. In cutoff calculations, aside from physical masses and momenta, such contributions are proportional to Λ^2, where Λ is the ultraviolet cutoff. In DR they must be proportional to suitable poles multiplied by μ^2, since this is the only available mass independent of the physical masses and momenta. If L is the number of loops, we have the condition $(\mu^{4-n})^L = \mu^2$, or $n = 4 - 2/L$. This means that quadratic divergences exhibit poles at $n = 2, 3, 10/3, \ldots$ for $L = 1, 2, 3, \ldots$ loop integrals. The same conclusion has been stated long ago by Veltman [8].

It should be pointed out that this is a useful criterion for scalar integrals of the form

$$I_{l,m} = \mathrm{i} \int \frac{d^n k}{(2\pi)^n} \frac{(k^2)^l}{(k^2 - M^2)^m}, \tag{6}$$

where l, m are integers ≥ 0 and for brevity we have not included the $\mathrm{i}\epsilon$ instruction. When l is a negative integer, there may appear poles at $n = 2$ which correspond to infrared, rather than ultraviolet singularities. A useful example is provided by the relation [10]:

$$\int \frac{d^n k}{k^2} = \int \frac{d^n k}{k^2 - M^2} - M^2 \int \frac{d^n k}{k^2(k^2 - M^2)}. \tag{7}$$

As is well known, the LHS is zero in DR. The first integral in the RHS is quadratically divergent in four dimensions and consequently exhibits an ultraviolet pole at $n = 2$, while the second one involves a pole at $n = 2$ arising from the Feynman parameter integration. This last singularity is related to the fact that the second integral contains a logarithmic infrared divergence at $n = 2$. Thus, in (7) we witness a cancellation between ultraviolet and infrared poles. As pointed out in [10], in discussing ultraviolet singularities one should include in that case the contribution from the first integral.

The above discussion can be extended to quartic divergences. Since, by an analogous argument, these must be proportional to μ^4, we have the relation $(\mu^{4-n})^L = \mu^4$, or $n = 4 - 4/L$. Thus, quartic divergences exhibit poles at $n = 0, 2, 8/3, \ldots$ for $L = 1, 2, 3, \ldots$ loop scalar integrals. Of course, as we will see in a specific example, a quartic divergence may also arise from the product of two one-loop quadratically divergent integrals, each of which has a pole at $n = 2$.

In Sect. 3, we show how these rules permit to establish a correspondence between DR and cutoffs calculations in one-loop amplitudes.

3 Regularization of $t_{\mu\nu}$

In order to implement a covariant regularization of $t_{\mu\nu}$, we first search for a four-dimensional representation of (4) and (5).

Inserting the well-known identity

$$\frac{1}{2\omega_k} = \int_{-\infty}^{\infty} dk_0 \, \delta(k^2 - m^2) \, \theta(k_0), \tag{8}$$

$k^2 \equiv k_0^2 - (\mathbf{k})^2$, in (4) and (5), we see that ρ and p are the zero–zero and i–i components of the formal tensor

$$t_{\mu\nu} = \int \frac{d^4 k}{(2\pi)^3} \, k_\mu k_\nu \, \delta(k^2 - m^2) \, \theta(k_0). \tag{9}$$

As we will discuss in detail later on, (9) can be regularized in the PV and DR frameworks. Since $t_{\mu\nu}$ is proportional to $g_{\mu\nu}$, in analogy with (2) and (3) it follows that

$$t_{\mu\nu} = \frac{g_{\mu\nu}}{4} \, m^2 \int \frac{d^4 k}{(2\pi)^3} \, \delta(k^2 - m^2) \, \theta(k_0), \tag{10}$$

where we have employed $k^2 \, \delta(k^2 - m^2) = m^2 \, \delta(k^2 - m^2)$. Using (8) in reverse and the identity

$$\frac{1}{\omega_k} = \frac{1}{\pi} \int_{-\infty}^{\infty} dk_0 \, \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\epsilon}, \tag{11}$$

which follows from contour integration, (10) can be cast in the form

$$t_{\mu\nu} = \frac{g_{\mu\nu}\ m^2}{4} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\epsilon}\,, \qquad (12)$$

which implies

$$t^\lambda{}_\lambda = m^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\epsilon}\,. \qquad (13)$$

We note that the integral in (12) and (13) is $\mathrm{i}\Delta_F(0)$, the Feynman propagator evaluated at $x = 0$.

An alternative derivation of (12) and (13) can be obtained by using the starting (4) and (5) to evaluate the trace $t^\lambda{}_\lambda$:

$$t^\lambda{}_\lambda = \frac{1}{2} \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\omega_k^2 - (\mathbf{k})^2}{\omega_k} = \frac{m^2}{2} \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{\omega_k}\,. \qquad (14)$$

Combining (3), (11) and (14), we immediately recover (12) and (13)! It is important to note that these expressions are proportional to m^2 and to the quadratically divergent integral $\mathrm{i}\Delta_F(0)$.

Returning to the issue of regularization, in the PV framework (9) is replaced by the regularized expression

$$(t_{\mu\nu})_{\mathrm{PV}} = \sum_{i=0}^{N} C_i \int \frac{\mathrm{d}^4k}{(2\pi)^3}\ k_\mu k_\nu\ \delta(k^2 - M_i^2)\ \theta(k_0)\,, \quad (15)$$

where $M_0 = m$, $C_0 = 1$, N is the number of regulator fields, M_j $(j = 1, 2, \ldots, N)$ denote their masses, and the C_i obey the constraints

$$\sum_{i=0}^{N} C_i\ (M_i^2)^p = 0 \quad (p = 0, 1, 2)\,. \qquad (16)$$

From (16) we see that in our case $N = 3$ is the minimum number of regulator fields. It is worthwhile to note that if the limit $M_j \to \infty$ is taken before the integration is carried out, (15) reduces to the original, unregularized expression of (9). In fact, $\lim_{M_j\to\infty}\ \delta(k^2 - M_j^2) = 0$. Following the steps leading from (9) to (12), (15) becomes

$$(t_{\mu\nu})_{\mathrm{PV}} = \frac{g_{\mu\nu}}{4} \sum_{i=0}^{N} C_i\ M_i^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\ \frac{\mathrm{i}}{k^2 - M_i^2 + \mathrm{i}\epsilon}\,, \quad (17)$$

which is the PV regularized version of (12).

The simplest way to evaluate (17) is to differentiate twice

$$I(M_i^2) \equiv \int \frac{\mathrm{d}^4k}{(2\pi)^4}\ \frac{\mathrm{i}}{k^2 - M_i^2 + \mathrm{i}\epsilon}\,, \qquad (18)$$

with respect to M_i^2, so that it becomes convergent. This leads to $I''(M_i^2) = 1/16\pi^2 M_i^2$. Integrating twice $I''(M_i^2)$ with respect to M_i^2, we obtain

$$I(M_i^2) = \frac{1}{16\pi^2}\left[M_i^2\ (\ln M_i^2 - 1) + K_1 M_i^2 + K_2\right]\,, \quad (19)$$

where K_1 and K_2 are arbitrary constants of integration. When inserted in (17), the terms involving K_1, K_2, and $-M_i^2$ cancel on account of (16), and (17) becomes

$$(t_{\mu\nu})_{\mathrm{PV}} = \frac{g_{\mu\nu}}{64\pi^2} \sum_{i=0}^{N} C_i\ M_i^4\ \ln\left(\frac{M_i^2}{\nu^2}\right). \qquad (20)$$

Here ν is a mass scale that can be chosen arbitrarily since its contribution vanishes on account of (16) with $p = 2$.

If we choose $N = 3$, the minimum number of regulator fields, the constants C_i can be expressed in terms of the M_i by solving (16) for $p = 0, 1, 2$ (see Appendix A). One finds that (20) contains three classes of contributions:
(i) terms quartic in the regulator masses M_j $(j = 1, 2, 3)$;
(ii) terms of $\mathcal{O}(m^2)$ which are quadratic in M_j;
(iii) terms of $\mathcal{O}(m^4)$.

As shown in (20), all these contributions are accompanied by logarithms. If one rescales the regulator masses by a common factor Λ, one finds that, modulo logarithms, the three classes become proportional to Λ^4, Λ^2, and Λ^0, respectively. Thus, the first class of terms exhibit the quartic divergence frequently invoked in discussions of the cosmological constant problem. However, as shown in the appendix, the results show a curious and at first hand unexpected feature: for arbitrary values of M_j, the sign of the quartic contribution to ρ is negative! Instead, the sign of the $\mathcal{O}(m^2\Lambda^2)$ term is positive, and that of $\mathcal{O}(m^4)$ contribution is negative.

The PV expression greatly simplifies in the limit $M_3 \to M_2 \to M_1 = \Lambda$ and becomes

$$(t_{\mu\nu})_{\mathrm{PV}} = \frac{g_{\mu\nu}}{128\pi^2}\left[-\Lambda^4 + 4m^2\Lambda^2 - m^4\left(3 + 2\ln\frac{\Lambda^2}{m^2}\right)\right]. \qquad (21)$$

It is interesting to note that (17) also follows from the PV regularization of the formal, quartically divergent tensor

$$J_{\mu\nu} = \mathrm{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\ \frac{k_\mu k_\nu}{k^2 - m^2 + \mathrm{i}\epsilon}\,. \qquad (22)$$

In fact

$$(J_{\mu\nu})_{\mathrm{PV}} = \mathrm{i}\sum_{i=0}^{N} C_i \int \frac{\mathrm{d}^4k}{(2\pi)^4}\ \frac{k_\mu k_\nu}{k^2 - M_i^2 + \mathrm{i}\epsilon}\,,$$

which, upon the replacement $k_\mu k_\nu \to g_{\mu\nu}k^2/4$ and the decomposition $k^2 = k^2 - M_i^2 + M_i^2$, reduces to (17), since the contribution of $k^2 - M_i^2$ vanishes on account of (16). Equation (22) may be also regulated by means of a Feynman regulator which, for this application, we choose to be of the form $[(\Lambda^2 - m^2)/(\Lambda^2 - k^2 - \mathrm{i}\epsilon)]^3$. Thus,

$$(J_{\mu\nu})_{\mathrm{FR}} = \mathrm{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\ \frac{k_\mu k_\nu}{k^2 - m^2 + \mathrm{i}\epsilon} \frac{(\Lambda^2 - m^2)^3}{(\Lambda^2 - k^2 - \mathrm{i}\epsilon)^3}\,. \quad (23)$$

Evaluating (23) we find the very curious result that it exactly coincides with (21)! An advantage of (23) is that one can discern immediately its sign by means of a Wick rotation of the k_0 axis. Replacing $k_\mu k_\nu \to g_{\mu\nu}k^2/4$, performing the rotation and introducing $k_0 = \mathrm{i}K_0$, we obtain the Euclidean representation

$$(J_{\mu\nu})_{\mathrm{FR}} = -\frac{g_{\mu\nu}}{4} \int \frac{\mathrm{d}^4K}{(2\pi)^4} \frac{K^2}{K^2+m^2} \frac{(\Lambda^2-m^2)^3}{(\Lambda^2+K^2)^3}\,, \quad (24)$$

which shows that the cofactor of $g_{\mu\nu}$ is manifestly negative, in conformity with (21).

In summary, according to (20) with the minimum number $N = 3$ of regulator fields, or its limit in (21), the leading contribution for a bosonic field would be $\rho = -\mathcal{O}(\Lambda^4)$, $p = \mathcal{O}(\Lambda^4)$! Such a result is theoretically unacceptable since for a scalar field

$$\langle 0|T_{00}|0\rangle = \frac{1}{2}\langle 0|\partial_0\varphi\partial_0\varphi + \partial_i\varphi\partial_i\varphi + m^2\varphi^2|0\rangle$$

should be positive. We therefore interpret the sign problem as an artifact of the regularization procedure that arises in the case $N = 3$ due to the fact that some of the regulator fields have negative metric.

The presence of quartic divergences, of either sign, has another highly unsatisfactory consequence, namely it breaks down the scale invariance of free field theories in the massless limit! We recall that the divergence of the dilatation current for a scalar field has the form

$$\partial_\mu D^\mu = T^\lambda{}_\lambda + \frac{\Box\phi^2}{2} = \Theta^\lambda{}_\lambda\,, \quad (25)$$

where $\Theta_{\mu\nu}$ is the "improved" energy-momentum tensor [11]. This leads to

$$\langle 0|\partial_\mu D^\mu|0\rangle = t^\lambda{}_\lambda = m^2\langle 0|\phi^2|0\rangle\,, \quad (26)$$

where we used $\Box\langle 0|\phi^2|0\rangle = 0$ and, in deriving the second equality in (26), we employed the equation of motion. Thus, for free fields, $t^\lambda{}_\lambda$ should vanish as $m \to 0$, a property that is violated by quartic divergences of either sign.

In order to circumvent the dual problems of sign and breakdown of scale invariance of the free field theory in the massless limit within the PV framework, there are two possibilities: one is to subtract the offending $\mathcal{O}(\Lambda^4)$ term in (21); the other is to employ $N \geq 4$, in which case the C_i are not determined by the M_i, and the sign of the $\mathcal{O}(\Lambda^4)$ contributions is undefined. In the last approach one can in principle impose the cancellation of the quartic divergence as a symmetry requirement. However, the sign of the leading $\mathcal{O}(m^2\Lambda^2)$ term remains undefined, which is not an attractive state of affairs.

A simpler and more satisfactory approach is to go back to (12) as a starting point to implement the regularization procedure. Regularization of (12) in the PV framework would lead us back to (17) and (21). Instead, we may regularize the integral in (12) with a Feynman regulator, which we choose to be of the form $[\Lambda^2/(\Lambda^2 - k^2 - \mathrm{i}\epsilon)]^2$. Neglecting terms of $\mathcal{O}(m^2/\Lambda^2)$, this leads to

$$(t_{\mu\nu})_{\mathrm{FR}} = \frac{g_{\mu\nu}}{64\pi^2}\left[m^2\Lambda^2 - m^4\left(\ln\frac{\Lambda^2}{m^2} - 1\right)\right]\,, \quad (27)$$

which implies

$$(t^\lambda{}_\lambda)_{\mathrm{FR}} = \frac{1}{16\pi^2}\left[m^2\Lambda^2 - m^4\left(\ln\frac{\Lambda^2}{m^2} - 1\right)\right]\,. \quad (28)$$

Equations (27) and (28) have the correct sign and conform with scale invariance in the massless limit! A similar result is obtained if a Wick rotation is implemented in (12), and an invariant cutoff is employed to evaluate the integral.

We now turn to DR. Since the steps from (9) to (12) involve only the k_0 integration, in DR the regularized expressions of (9) and (12) are equivalent and we obtain

$$(t_{\mu\nu})_{\mathrm{DR}} = \frac{g_{\mu\nu}\ m^2\ \mu^{(4-n)}}{n} \int \frac{\mathrm{d}^nk}{(2\pi)^n}\frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\epsilon}\,, \quad (29)$$

which leads to

$$(t^\lambda{}_\lambda)_{\mathrm{DR}} = m^2\ \mu^{(4-n)} \int \frac{\mathrm{d}^nk}{(2\pi)^n}\frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\epsilon}\,. \quad (30)$$

We note that the $n = 0$ pole in (29) arises from the replacement $k_\mu k_\nu \to g_{\mu\nu}k^2/n$. Since $g^{\mu\nu}g_{\mu\nu} = n$, this pole is absent in the evaluation of the trace in (30). In order to use the rules of correspondence in an unambiguous manner, we apply them to the Lorentz scalar $t^\lambda{}_\lambda$ evaluated in the FR and DR frameworks. Carrying out the integration in (30), we find

$$(t^\lambda{}_\lambda)_{\mathrm{DR}} = \frac{4\ m^4}{(2\sqrt{\pi})^n}\frac{(\mu/m)^{(4-n)}\ \Gamma(3-n/2)}{(2-n)(4-n)}\,. \quad (31)$$

This expression exhibits poles at $n = 4$ and $n = 2$ which, according to the rules of correspondence for one-loop integrals, indicate the presence of logarithmic and quadratic divergences.

A heuristic way to establish a correspondence between the cutoff calculation in (28) and the DR expression in (31), is to carry out the expansion about $n = 4$ in the usual way, but at the same time separate out the $n = 2$ pole in such a manner that the overall result is only modified in $\mathcal{O}(n-4)$. This leads to

$$(t^\lambda{}_\lambda)_{\mathrm{DR}} = \frac{\mu^2m^2}{2\pi}\left[\frac{1}{2-n} + \frac{1}{2}\right] - \frac{m^4}{16\pi^2}\left[\frac{2}{4-n} + \ln\frac{\mu^2}{m^2} - 2C + 1\right] + \mathcal{O}(n-4)\,, \quad (32)$$

where $C = [\gamma - \ln 4\pi]/2$. The contribution proportional to m^4 represents the usual result. The first term contains the pole at $n = 2$, and only modifies the expansion in $\mathcal{O}(n-4)$. A correspondence with (28) can be implemented by means of the identifications

$$\left[\frac{1}{4-n} + \ln\frac{\mu}{m} - C\right]_{n\approx 4} \to \ln\frac{\Lambda}{m} - 1\,, \quad (33)$$

$$\frac{\mu^2}{2\pi}\left[\frac{1}{2-n} + \frac{1}{2}\right]_{n\approx 2} \to \frac{\Lambda^2}{16\pi^2}\,, \quad (34)$$

where, for instance, $n \approx 2$ means that n is in the immediate neighborhood of 2. It is interesting to note that if one approaches the ultraviolet poles from below, as it seems natural in DR, the signs of the left and right sides of (33) and (34) coincide!

Another piece of interesting information contained in the DR expression of (31) is that the $\mathcal{O}(m^2\Lambda^2)$ contribution is not accompanied by a $\ln(\Lambda^2/m^2)$ cofactor. This can be seen as follows: since (31) is proportional to $(m^2)^{n/2}$, if we differentiate twice with respect to m^2 we see that the pole at $n = 2$ disappears. As a consequence, terms of $\mathcal{O}(m^2\Lambda^2 \ln(\Lambda^2/m^2))$ cannot be present, since otherwise contributions of $\mathcal{O}(\Lambda^2)$ would survive under the double differentiation. Indeed, this observation agrees with (28). Thus, in one-loop calculations depending on m^2, Λ^2, terms of $\mathcal{O}(m^2\Lambda^2 \ln(\Lambda^2/m^2))$ would require a double pole at $n = 2$ in the DR expression.

A conclusion essentially identical to (27), namely that the divergence of the zero-point energy for free particles is quadratic rather then quartic, and that massless particles do not contribute, has been recently advocated by Akhmedov [5], invoking arguments of relativistic invariance. The analysis of the present paper shows that this is not enough to single out (27), since the PV regularization leads to the covariant expressions of (20) and (21) that exhibit a quartic divergence. What singles out (27) are the combined requirements of relativistic covariance and scale invariance of free field theories in the massless limit, as well as consistency with the rules of correspondence.

We conclude this section by recalling that the contributions of all bosons (fermions) carry the same (opposite) sign as (27). Each contribution must be multiplied by a factor η that takes into account the color and helicity degrees of freedom, as well as the particle–antiparticle content.

4 Evaluation of $t^\lambda{}_\lambda$ based on Feynman diagrams

In Sect. 3 we have discussed the regularization of $t_{\mu\nu}$ and its trace in the free field theory case, starting from the familiar expressions for ρ and p given in (4) and (5). It is instructive to revisit the evaluation of $t^\lambda{}_\lambda$ on the basis of well-known expressions for the vacuum expectation value of products of free field operators on the one hand, and Feynman diagrams on the other. This will also pave the way to the discussion of the effect of interactions in Sect. 5.

We will consider three examples: an hermitian scalar field, a spinor field, and a vector boson, all endowed with mass m. We recall the free field theory expressions for $T^\lambda{}_\lambda$ in the three cases:

$$T^\lambda{}_\lambda = -\partial_\lambda\varphi\partial^\lambda\varphi + 2m^2\varphi^2 \,, \tag{35}$$

$$T^\lambda{}_\lambda = -3\bar{\psi}\left[\mathrm{i}\frac{\overset{\leftrightarrow}{\partial\!\!\!/}}{2} - m\right]\psi + m\bar{\psi}\psi \,, \tag{36}$$

$$T^\lambda{}_\lambda = -m^2 A_\lambda A^\lambda \,. \tag{37}$$

The above formulae are valid in four dimensions and, in deriving (37), we have employed the symmetric version of $T_{\mu\nu}$ for the spin 1 field.

A direct way of evaluating $t^\lambda{}_\lambda$ is to consider the vacuum expectation value of two fields at x and y, carry out the differentiations exhibited in (35) and (36) and take the limit $x \to y$. For instance, in the free field scalar case we have the well-known representation:

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^3}\,\delta(k^2 - m^2)\,\theta(k_0)\,\mathrm{e}^{-\mathrm{i}k(x-y)} \,. \tag{38}$$

The RHS of (38), the $\mathrm{i}\Delta^+(x-y)$ function, is the contribution of the one-particle intermediate state in the Källen–Lehmann representation which, of course, is the only one that survives in the free field theory case. From (38) we find

$$\begin{aligned}&\langle 0| - \partial_\lambda\varphi(x)\partial^\lambda\varphi(y) + 2m^2\varphi(x)\varphi(y)|0\rangle \qquad (39)\\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^3}\,\delta(k^2 - m^2)\,\theta(k_0)\,(2m^2 - k^2)\,\mathrm{e}^{-\mathrm{i}k(x-y)} \,,\end{aligned}$$

which is well defined. Taking the limit $x \to y$ and recalling (35), we obtain the formal expression

$$(t^\lambda{}_\lambda)_\varphi = \int \frac{\mathrm{d}^4k}{(2\pi)^3}\,\delta(k^2 - m^2)\,\theta(k_0)\,(2m^2 - k^2) \,. \tag{40}$$

If instead of $T_{\mu\nu}$, the "improved" tensor $\Theta_{\mu\nu}$ [11] is employed for scalar fields, (35) is replaced by $\Theta^\lambda{}_\lambda = \varphi\Box\varphi + 2m^2\varphi^2$, which again leads to (39) and (40). If we replace $k^2 \to m^2$ in these expressions on account of $\delta(k^2 - m^2)$, we recover (10), the result of our previous analysis in Sect. 3. Parenthetically, we recall that, at the classical level, use of the equations of motion leads to $\Theta^\lambda{}_\lambda = m^2\varphi^2$ even in the presence of the $\lambda\varphi^4$ interaction [11].

Equation (40) admits another representation that can be linked with a Feynman vacuum diagram, to wit

$$(t^\lambda{}_\lambda)_\varphi = \mathrm{Re}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{i}\,\left[2m^2 - k^2\right]}{k^2 - m^2 + \mathrm{i}\epsilon} \,, \tag{41}$$

where we have employed $\pi\delta(k^2 - m^2) = \mathrm{Re}(\mathrm{i}/k^2 - m^2 + \mathrm{i}\epsilon)$ and used the fact that the integrand is even in k_0 to replace $\theta(k_0) \to 1/2$.

Equivalently, we have

$$(t^\lambda{}_\lambda)_\varphi = \mathrm{Re}\left\{m^2\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\epsilon} - \mathrm{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\right\} . \tag{42}$$

Equation (41) is depicted in Fig. 1, where the cross indicates the insertion of the operator $T^\lambda{}_\lambda$ given in (35). We note that the "Re" instruction is important in the passage from (40) to (41) and ensures that the answer is real, as required for diagonal matrix elements of the hermitian operator $T^\lambda{}_\lambda$.

Using (36), the corresponding expression in the fermion case is

$$(t^\lambda{}_\lambda)_\psi = -\mathrm{Re}\,\mathrm{Tr}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{i}\,[m - 3(k\!\!\!/ - m)]}{k\!\!\!/ - m + \mathrm{i}\epsilon} \,. \tag{43}$$

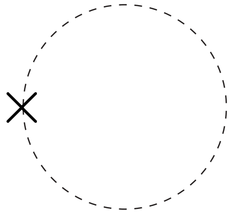

Fig. 1. One-loop vacuum amplitudes. The cross indicates the insertion of the trace ${T^\lambda}_\lambda$ of the energy-momentum tensor. The dashed line represents a scalar, spinor or massive vector particle (see Sect. 4)

This can be cast in the form

$$({t^\lambda}_\lambda)_\psi = \tag{44}$$
$$- \operatorname{Re}\left\{4\, m^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\epsilon} - 12\,\mathrm{i} \int \frac{\mathrm{d}^4k}{(2\pi)^4}\right\},$$

where we have employed $\int \mathrm{d}^4k\, \not{k}/(k^2 - m^2 + \mathrm{i}\epsilon) = 0$.

Finally, using (37), we have

$$({t^\lambda}_\lambda)_A = \operatorname{Re}\left\{3m^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\epsilon} - \mathrm{i} \int \frac{\mathrm{d}^4k}{(2\pi)^4}\right\}. \tag{45}$$

These expressions exhibit interesting features: the first terms in (42), (44) and (45) are quadratically divergent and real. The reality condition is easily checked by performing the k_0 contour integration or by means of a Wick rotation of the k_0 axis accompanied by the change of variable $k_0 = \mathrm{i}K_0$. This rotation is mathematically allowed since the k_0 integrations in those contributions are convergent. The second terms in (42), (44) and (45) formally exhibit a quartic divergence. However, since the integrations are over the real axes, such terms are purely imaginary in Minkowski space and therefore do not contribute if the "Re" restriction is imposed. We note parenthetically that, unlike in the previous case, the Wick rotation cannot be applied to the unregularized $\int \mathrm{d}^4k$ as it stands since, in performing the k_0 integration, the contributions of the large quarter circles in the complex plane are not negligible and, in fact, they are necessary to satisfy Cauchy's theorem.

In the PV and DR approaches the regularized versions of the imaginary contributions in (42), (44) and (45) vanish automatically. In contrast, a Feynman regulator of the form $[\Lambda^2/(\Lambda^2 - k^2 - \mathrm{i}\epsilon)]^3$ leads to

$$-\mathrm{i} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \left(\frac{\Lambda^2}{\Lambda^2 - k^2 - \mathrm{i}\epsilon}\right)^3 = \frac{\Lambda^4}{32\pi^2}, \tag{46}$$

which is real and positive and exhibits the frequently invoked quartic divergence. The reality property can also be checked by performing a Wick rotation in (46), which is now mathematically allowed. Thus, we see that the quartically divergent contributions in the one-loop vacuum diagrams have a very ambivalent and disturbing property: their contributions to ρ are imaginary in Minkowski space and real, if regularized according to (46), in Euclidean space.

However, if (46) is applied to regularize the imaginary parts between curly brackets in (42), (44) and (45), serious inconsistencies emerge. In fact, their coefficients do not conform with the relations $({t^\lambda}_\lambda)_\psi = -4\ ({t^\lambda}_\lambda)_\varphi$ and $({t^\lambda}_\lambda)_A = 3\ ({t^\lambda}_\lambda)_\varphi$, which arise on account of the helicity and particle–antiparticle degrees of freedom of Dirac spinors and massive vector bosons in four dimensions.

A direct way to see that these terms are inconsistent with (40) is to go back to that expression, replace $k^2 \to m^2$ on account to the δ-function and then use $\delta(k^2 - m^2) = (1/\pi)\operatorname{Re}(\mathrm{i}/k^2 - m^2 + \mathrm{i}\epsilon)$. This leads to the first term of (42), a result that is only consistent with (40) if the second contribution vanishes.

We conclude that, in order to avoid inconsistencies, the quartically divergent imaginary parts in (42), (44) and (45) must be subtracted either by imposing the reality condition in Minkowski space or by means of the regularization procedure, as in the DR and PV cases. The surviving terms in (42), (44) and (45) are proportional to $m^2 \mathrm{i}\Delta_F(0)$, satisfy the relations $({t^\lambda}_\lambda)_\psi = -4\ ({t^\lambda}_\lambda)_\varphi$ and $({t^\lambda}_\lambda)_A = 3\ ({t^\lambda}_\lambda)_\varphi$, and coincide with the result in (40) and its equivalent expression in (10). Furthermore, they conform with the scale invariance of free field theories in the massless limit.

The PV, DR, and FR regularizations and their correspondence was discussed in detail in Sect. 3 in the case of the free scalar field, starting with (12) and (13). In writing down the rules of correspondence between DR and four-dimensional cutoff calculations in the case of spin 1 and spin 1/2 fields, there is a subtlety that should be pointed out. In the case of the spin 1 field, the DR version of ${t^\lambda}_\lambda$ is given by the expression for the scalar field (cf. (30)) multiplied by $n-1$, the number of helicity degrees of freedom in n dimensions. In separating out the contribution of the $n = 2$ pole (cf. (32)), the residue carries then a factor 1, rather than 3. In order to maintain the proper relation with the four-dimensional calculation, in the spin 1 case the LHS of (34) corresponds to $3\Lambda^2/16\pi^2$ rather than $\Lambda^2/16\pi^2$, the factor 3 reflecting the number of helicity degrees of freedom in four dimensions. A similar rule holds for spin 1/2 fields: if in evaluating the $n = 2$ residue one employs $\mathrm{Tr}\ \mathbb{1} = 2$, as befits a spinor in two dimensions, in the rule of correspondence with the four-dimensional cutoff calculation one includes an additional factor 2 to reflect the fact that $\mathrm{Tr}\ \mathbb{1} = 4$ for four-dimensional spinors.

The possible dichotomy in the treatment of the helicity degrees of freedom has had an interesting effect in the derivation of sum rules based on the speculative assumption that one-loop quadratic divergences cancel in the standard model (SM). As explained in [10], in DR the condition of cancellation of quadratic divergences in one-loop tadpole diagrams is given by

$$\mathrm{Tr}\ \mathbb{1}\ \sum_f m_f^2 = 3\ m_H^2 + (2\ m_W^2 + m_Z^2)(n-1), \tag{47}$$

where the f summation is over fermion masses and includes the color degree of freedom. The factor $n-1$ reflects once more the helicity degrees of freedom of spin 1 bosons in

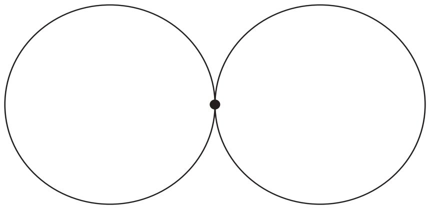

Fig. 2. Two-loop vacuum amplitude in $\lambda\phi^4$ theory (see Sect. 5)

n dimensions. Equation (47) leads also to the cancellation of all quadratic divergences in the one-loop contributions to the Higgs boson and fermion self-energies. Setting $n = \mathrm{Tr}\ \mathbb{1} = 4$, and neglecting the contributions of the lighter fermions one obtains the Veltman–Nambu sum rule [8, 9]:

$$m_H^2 = 4\ m_t^2 - 2\ m_W^2 - m_Z^2\,, \tag{48}$$

On the other hand, it was pointed out in [10] that in DR (47) with $n = 4$ is not sufficient to cancel the remaining quadratic divergences in the W and Z self-energies. Associating once more the one-loop quadratic divergences with the $n = 2$ poles, the cancellation of the residues in all cases (f, H, W, Z) takes place when $n = 2$ is chosen. With $n = 2$ and $\mathrm{Tr}\ \mathbb{1} = 2$ [12], this leads to the alternative sum rule [10]

$$m_H^2 = 2\ m_t^2 - \frac{(2\ m_W^2 + m_Z^2)}{3}\,. \tag{49}$$

Inserting the current values, $m_t = 174.3\,\mathrm{GeV}$, $m_Z = 91.1875\,\mathrm{GeV}$, and $m_W = 80.426\,\mathrm{GeV}$ [13], (48) and (49) lead to the predictions $m_H = 317\,\mathrm{GeV}$, and $m_H = 232\,\mathrm{GeV}$, respectively. The current 95% CL upper bound from the global fit to the SM is $m_H^{95} = 211\,\mathrm{GeV}$ [13], so that the above values are somewhat larger than the range favored by the electroweak analysis. It will be interesting to see whether these predictions ultimately bear any relation to reality!

5 Effect of interactions

An important issue is what happens when interactions are taken into account. The investigation of their effect on ${t^\lambda}_\lambda$ is an open and difficult one, since vacuum matrix elements are factored out and then cancelled in the usual treatment of quantum field theory. As it is well known, in the conventional framework, interactions break the scale invariance of free field theories in the massless limit, a phenomenon referred to as the trace anomaly [14–16]. One naturally expects that a similar phenomenon takes place in vacuum amplitudes, an occurrence that would lead to the emergence of quartic divergences. In this section we limit our analysis to two instructive examples.

We first discuss the question in the scalar theory with $\mathcal{L}_{\mathrm{int}} = -(\lambda/4!)\phi^4$. One readily finds that in $\mathcal{O}(\lambda)$ this interaction contributes $-(\lambda/8)\Delta_{\mathrm{F}}^2(0)$ to ρ, which is quartically divergent. This result is obtained by either using the familiar plane wave expansion involving annihilation and creation operators and their commutation relations,

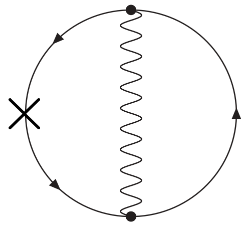

Fig. 3. Two-loop vacuum amplitude in QED. The cross represents the insertion of $-(n-1)\bar\psi\left(\mathrm{i}\ \overset{\leftrightarrow}{\partial\!\!\!/}\ /2 - m\right)\psi$ (see Sect. 5)

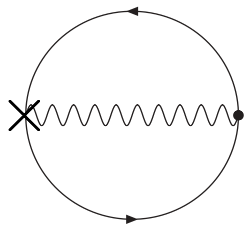

Fig. 4. Two-loop vacuum amplitude in QED. The cross represents the insertion of $(n-1)e\bar\psi A\!\!\!/\psi$ (see Sect. 5)

or by the calculation of the relevant Feynman diagram, which is a "figure 8" with the interaction at the intersection (see Fig. 2). We note that $1/8 = 3/4!$ is the symmetry number for this diagram. However, the mass in (12) is the unrenormalized mass m_0, since this is the parameter that appears in the Lagrangian. Writing $m_0^2 = m^2 - \Pi(m^2)$, where Π is the self-energy, (12) generates a counterterm $-\Pi(m^2)\mathrm{i}\Delta_{\mathrm{F}}(0)/4$. In $\mathcal{O}(\lambda)$ the only contribution to $\Pi(m^2)$ is the seagull diagram and equals $(\lambda/2)\mathrm{i}\Delta_{\mathrm{F}}(0)$, where $1/2$ is the symmetry number. Thus, the counterterm $(\lambda/8)\Delta_{\mathrm{F}}^2(0)$ exactly cancels the quartic divergence from the "figure 8" diagram! The same cancellation occurs when regulator fields are present, since (17) involves just a linear combination of terms analogous to (12)!

Next we consider the example of QED in $\mathcal{O}(e^2)$. Choosing the symmetric and explicitly gauge invariant version of $T_{\mu\nu}$, one finds in n dimensions:

$$T^\lambda{}_\lambda = \frac{n-4}{4}\ F_{\mu\nu}\ F^{\mu\nu} - (n-1)\bar\psi\left[\mathrm{i}\frac{\overset{\leftrightarrow}{D\!\!\!\!/}}{2} - m_0\right]\psi + m_0\ \bar\psi\ \psi\,, \tag{50}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu + \mathrm{i}eA_\mu$ is the covariant derivative and ψ and A_μ are the electron and photon fields. It is easy to see that the insertion of $-(n-1)\bar\psi(\mathrm{i}\ \overset{\leftrightarrow}{\partial\!\!\!/}\ /2 - m_0)\psi$ in the electron loop, corrected by the interaction in order e^2 (Fig. 3), cancels against the contribution of $(n-1)e\bar\psi A\!\!\!/\psi$ corrected in $\mathcal{O}(e)$ (see Fig. 4). This reflects the validity of the equation of motion $(\mathrm{i}D\!\!\!\!/ - m_0)\ \psi = 0$.

Next, we focus on the insertion of $[(n-4)/4]F_{\mu\nu}F^{\mu\nu}$ in the photon line (Fig. 5). The fermion-loop integral with two external off-shell photons of momentum q is given by

$$\frac{-\mathrm{i}8e^2}{(2\sqrt{\pi})^n}\ \mu^{(4-n)}\ \Gamma(2-n/2)\ (g_{\mu\nu}q^2 - q_\mu q_\nu)$$

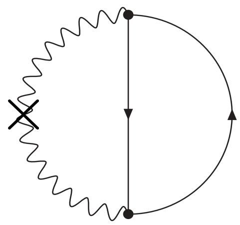

Fig. 5. Two-loop vacuum amplitude in QED. The cross represents the insertion of $[(n-4)/4]F_{\mu\nu}F^{\mu\nu}$ (see Sect. 5)

$$\times \int_0^1 \mathrm{d}x\; x\;(1-x)\;\left[m^2 - q^2 x(1-x)\right]^{\frac{n}{2}-2}, \quad (51)$$

where m is the fermion mass. Closing the photon line and inserting the vertex $[(n-4)/4]F_{\mu\nu}F^{\mu\nu}$ multiplies (51) by

$$\mu^{(4-n)} \int \frac{\mathrm{d}^n q}{(2\pi)^n} \frac{n-4}{2}\; q^2 g^{\mu\nu} \left(\frac{-\mathrm{i}}{q^2}\right)^2$$

and we obtain for the diagram of Fig. 5

$$\mathcal{A}_{\mathrm{DR}} = \frac{4\mathrm{i}e^2}{(2\sqrt{\pi})^n}(\mu^{4-n})^2(n-4)(n-1)\Gamma(2-n/2) \quad (52)$$
$$\times \int_0^1 \mathrm{d}x\; x\;(1-x) \int \frac{\mathrm{d}^n q}{(2\pi)^n}\left[m^2 - q^2 x(1-x)\right]^{\frac{n}{2}-2}.$$

We note that $(1/q^2)^2$, arising from the two photon propagators in Fig. 5, has been cancelled by two q^2 factors, one from (51), the other from the $[(n-4)/4]F_{\mu\nu}F^{\mu\nu}$ vertex.

Performing a Wick rotation of the q_0 axis, and introducing

$$q_0 = \mathrm{i}Q_0\,, \quad \mathbf{q} = \mathbf{Q}\,, \quad \mathrm{d}^n Q = \frac{\pi^{n/2}}{\Gamma(n/2)}\; Q^{2\left(\frac{n-2}{2}\right)}\mathrm{d}Q^2\,,$$
$$u = Q^2\,,$$

we have

$$\mathcal{A}_{\mathrm{DR}} = -e^2 C(n) \int_0^1 \mathrm{d}x\,[x(1-x)]^{\frac{n}{2}-1}$$
$$\times \int_0^\infty \mathrm{d}u\; u^{\frac{n}{2}-1}\left[u + \frac{m^2}{x(1-x)}\right]^{\frac{n}{2}-2}, \quad (53)$$

where

$$C(n) = \frac{4}{(4\pi)^n}\;\frac{\Gamma(2-n/2)}{\Gamma(n/2)}(n-4)(n-1)(\mu^{4-n})^2\,.$$

The u-integral in (53) equals

$$[\Gamma(n/2)\Gamma(2-n)/\Gamma(2-n/2)]\;\; m^{2n-4}\,[x(1-x)]^{2-n}\,,$$

and (53) becomes

$$\mathcal{A}_{\mathrm{DR}} = -\frac{4\,e^2}{(4\pi)^n}(n-4)(n-1)(\mu^{4-n})^2\;\Gamma(2-n)m^{2n-4}$$
$$\times \int_0^1 \mathrm{d}x\,[x(1-x)]^{1-n/2}\,. \quad (54)$$

The remaining integral equals $B(2-n/2, 2-n/2) = \Gamma^2(2-n/2)/\Gamma(4-n)$ and we find

$$\mathcal{A}_{\mathrm{DR}} = \frac{16\;e^2\;m^4\;(\mu/m)^{2(4-n)}(n-1)\;\Gamma^2(3-n/2)}{(4\pi)^n(2-n)(3-n)(4-n)}\,. \quad (55)$$

Equation (55) exhibits simple poles at $n = 2, 3, 4$. According to the rules of correspondence for two-loop diagrams, this indicates the presence of quartic, quadratic, and logarithmic divergences.

For clarity, we point out that potentially there is a counterterm diagram associated with Fig. 5, in which the fermion loop is replaced by the insertion of the field renormalization vertex $-\mathrm{i}\delta Z F_{\mu\nu}F^{\mu\nu}/4$ in the closed photon loop. However, such diagram, involving two vertices proportional to $F_{\mu\nu}F^{\mu\nu}$ and two photon propagators, leads to a result proportional to $\mathrm{i}\int \mathrm{d}^n q$, which is imaginary in Minkowski space and furthermore vanishes in DR. In particular, this also means that if the result for the fermion-loop subintegration given in (51) were expanded about $n = 4$, the pole contribution would cancel when the q integration is performed. Since we need the full dependence on n to determine the possible pole positions and the pole contribution from the fermion-loop subintegration vanishes, in (55) we have evaluated the full two-loop integral, without expanding the fermion-loop subintegration about $n = 4$.

There remains the contribution of $m_0\bar{\psi}\psi$, the last term in (50), corrected by the interaction in order e^2. The superficial degree of divergence in four dimensions of the corresponding two-loop diagram is trilinear, so that one expects a quadratic divergence. Indeed, the DR calculation of the diagram shows a pole at $n = 3$ and a double pole at $n = 4$ which, according to the rules of correspondence for two-loop diagrams, indicate a quadratic divergence and logarithmic singularities proportional to $\mathcal{O}(m^4 \ln(\Lambda^2/m^2))$ and $\mathcal{O}(m^4 \ln^2(\Lambda^2/m^2))$.

6 Conclusions

In this paper we discuss a number of issues related to the nature of the contributions of fundamental particles to the vacuum energy density. This problem is of considerable conceptual interest since what may be called the physics of the vacuum is not addressed in the usual treatment of quantum field theory. On the other hand, it also represents a major unsolved problem since estimates of these contributions show an enormous mismatch with the observed cosmological constant.

As a preamble to our analysis, in Sect. 2 we use an elementary argument to derive rules of correspondence between the poles' positions in DR and ultraviolet cutoffs in four-dimensional calculations. In the case of quadratic divergences, they coincide with Veltman's dictum [8], while they are extended here to quartic singularities. A specific example of this correspondence is given at the one-loop level in Sect. 3.

In Sect. 3 we address the Lorentz-covariant regularization of $t_{\mu\nu}$ in free- field theories, starting from the elementary expressions for ρ and p. Making use of a mathematical identity, we are led to a covariant expression, which is immediately confirmed by the direct evaluation of $t^\lambda{}_\lambda$. The regularization of this result is then implemented in the PV, DR, and FR frameworks. In Sect. 4, we re-examine $t_{\mu\nu}$ on the basis of the well-known expression for the vacuum expectation value of products of free fields, as well as one-loop Feynman vacuum diagrams, with results that are consistent with those of Sect. 3. In Sect. 5, we consider two cases involving interactions: $\lambda\phi^4$ theory in $\mathcal{O}(\lambda)$ and QED in $\mathcal{O}(e^2)$, which require the examination of two-loop vacuum diagrams.

Our general conclusion, based on Lorentz covariance and the scale invariance of free field theories in the massless limit, as well as consistency with the rules of correspondence applied to $t^\lambda{}_\lambda$, is that quartically divergent contributions to ρ are absent in the case of free fields.

At first hand, the notion that free photons do not contribute to ρ may seem strange. However, we point out that this immediately follows from (50), which tells us that for free photons $(T^\lambda{}_\lambda)_\gamma = 0$ in four dimensions. This implies $(t^\lambda{}_\lambda)_\gamma = 0$ and, using (2) and (3), $\rho_\gamma = 0$! In more pictorial language: $(T^\lambda{}_\lambda)_\gamma = 0$ implies $\rho_\gamma = 3p_\gamma$, the equation of state of a photon gas but, in the vacuum case, (2) tells us that $\rho^{\text{vac}} = -p^{\text{vac}}$ for any field. The only way of satisfying the two constraints in the vacuum case is $\rho^{\text{vac}}_\gamma = p^{\text{vac}}_\gamma = 0$.

As pointed out in Sect. 3, in the case of free fields the same conclusion was recently advocated in the interesting work of Akhmedov [5] on the basis of a less complete line of argumentation, and without examining the effect of interactions.

When interactions are turned on, as illustrated in the QED case in Sect. 5, our conclusion is that quartic divergences generally emerge. Thus, in some sense there is a parallelism between the analysis of vacuum amplitudes and conventional quantum field theory. Free-field theories are scale invariant in the massless limit and, according to our interpretation, this partial symmetry protects the theory from the emergence of quartic divergences. However, in the presence of interactions, the symmetry is broken even in the massless limit and consequently such singularities generally arise. On the other hand, it is worthwhile to recall that $T^\lambda{}_\lambda$ becomes a soft operator even in the presence of interactions under the speculative assumption that the coupling constants are zeros of the relevant β-functions [15, 16].

From the point of view of formal renormalization theory, the presence of the highly divergent expressions encountered in the study of vacuum amplitudes does not present an insurmountable difficulty. For instance, in the PV approach discussed in Sect. 3 with a sufficiently large number of regulator fields, the coefficients of the free field divergences are undefined and in principle can be chosen to cancel the corresponding singularities emerging from interactions. More generally, the λ constant that appears in Einstein's equation can be adjusted to cancel such singularities. As emphasized by several authors [1] the crisis resides in the extraordinarily unnatural fine-tuning that these cancellations entail.

From a practical point of view, the conclusions in the present paper hardly affect the cosmological constant problem: clearly, it makes very little difference phenomenologically whether the mismatch is 123 or 120 orders of magnitude! On the other hand, they place the origin of the problem on a different conceptual basis.

The simplest framework in which quartic divergences cancel remains supersymmetry since it implies an equal number of fermionic and bosonic degrees of freedom. In some effective supergravity theories derived from four-dimensional superstrings, with broken supersymmetry, it is possible to ensure also the cancellation of the $\mathcal{O}(m^2\Lambda^2)$ terms in the one-loop effective potential [17]. In such scenarios, the $\mathcal{O}(m^4)$ terms become $\mathcal{O}(m^4_{3/2})$, where $m_{3/2}$, the gravitino mass, is associated with the scale of supersymmetry breaking. Assuming $m_{3/2} = \mathcal{O}(1\,\text{TeV})$ this leads to the rough estimate $\rho = \mathcal{O}(\text{TeV}^4)$ mentioned in the Introduction.

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Appendix A

In this appendix we analyze (20), which is the PV-regularized version of (12). Choosing the minimum number $N = 3$ of regulator fields, the constants C_j $(j = 1, 2, 3)$ can be expressed in terms of the m^2 and the M_j^2 $(j = 1, 2, 3)$ by solving (16) for $p = 0, 1, 2$. This leads to

$$C_1 = \frac{-M_2^2M_3^2 + m^2(M_2^2 + M_3^2) - m^4}{(M_2^2 - M_1^2)(M_3^2 - M_1^2)}\,. \tag{A1}$$

C_2 is obtained from C_1 by applying the cyclic permutation (1 2 3), while C_3 is obtained from C_2 by means of the same permutation. Focusing on the quartic divergences, we neglect the terms proportional to m^2 and m^4 and,since ν^2 in (20) is arbitrary, we choose $\nu^2 = M_3^2$. Then the quartically divergent terms in (20) can be cast in the form

$$(t_{\mu\nu})_{\text{PV}}(m^2 = 0) = \frac{g_{\mu\nu}}{64\pi^2}\frac{M_1^2M_2^2M_3^2}{M_2^2 - M_1^2}\left[f\left(\frac{M_3^2}{M_1^2}\right) - f\left(\frac{M_3^2}{M_2^2}\right)\right]\,, \tag{A2}$$

where

$$f(x) = \frac{\ln x}{x-1}\,. \tag{A3}$$

We note that $f(x)$ is positive definite for all $x \geq 0$ while its derivative

$$f'(x) = \frac{1}{x-1}\left[\frac{1}{x} - \frac{\ln x}{x-1}\right] \tag{A4}$$

is negative definite. Thus, $f(x)$ is a positive definite and decreasing function of its argument. Consider now the case $M_2^2 > M_1^2$. Then $M_3^2/M_1^2 > M_3^2/M_2^2$ and the expression between square brackets is negative. Since $(M_2^2 - M_1^2) > 0$, we conclude that the coefficient of $g_{\mu\nu}$ in (A2) is negative for any value of M_3^2. If $M_1^2 > M_2^2$, $M_3^2/M_1^2 < M_3^2/M_2^2$, the square bracket is positive but $(M_2^2 - M_1^2) < 0$, so that we reach the same conclusion. Thus, for all possible values of the regulator masses M_j^2, the coefficient of $g_{\mu\nu}$ in (20) is negative definite which, as explained in Sect. 3, is physically unacceptable. Analogous arguments show that for all M_j^2, the contributions of $\mathcal{O}(m^2 M_j^2)$ and $\mathcal{O}(m^4)$ in the cofactor of $g_{\mu\nu}$ are positive and negative definite, respectively. In the limit $M_3 \to M_2 \to M_1 = \Lambda$, (A2) greatly simplifies and reduces to the first term in (21).

As mentioned in Sect. 3, one possible solution of the sign problem is to subtract the offending $\mathcal{O}(M^4)$ contributions. Another possibility is to consider $N \geq 4$ regulator fields. In that case (16) for $p = 0, 1, 2$ are not sufficient to determine the C_j $(j = 1, \ldots, N)$ in terms of the masses. As expected, we have checked that the coefficient of the $\mathcal{O}(M^4)$ term in (20) becomes undetermined while still satisfying the three relations of (16), so that it may be chosen to be positive or, for that matter, zero. The last possibility would naturally follow by invoking the scale invariance of free field theories in the massless limit and would conform with the analysis based on DR. As mentioned in Sect. 3, in the $N \geq 4$ solution of the problem, the coefficient of the leading $\mathcal{O}(m^2 \Lambda^2)$ is also undefined, which is not a satisfactory state of affairs!

Appendix B

In this appendix we apply DR and the rules of correspondence to integrals that occur in the evaluation of the one-loop effective potential in the $\lambda\phi^4$ theory [18]:

$$V(\phi_c) = -\frac{\mathrm{i}\,\mu^{4-n}}{2}\int \frac{\mathrm{d}^n k}{(2\pi)^n} \ln\left(\frac{k^2 - m^2 - \frac{1}{2}\lambda\phi_c^2 + \mathrm{i}\epsilon}{k^2 - m^2 + \mathrm{i}\epsilon}\right). \tag{B1}$$

We recall that in the path integral formalism, the denominator in the argument of the logarithm arises from the normalization of the generating functional $W[J]$, namely $W[0] = 1$.

We consider the integral

$$K = -\frac{\mathrm{i}\,\mu^{4-n}}{2}\int \frac{\mathrm{d}^n k}{(2\pi)^n} \ln(k^2 - c + \mathrm{i}\epsilon)\,, \tag{B2}$$

which can be obtained from

$$L = -\frac{\mathrm{i}\,\mu^{4-n}}{2}\int \frac{\mathrm{d}^n k}{(2\pi)^n} (k^2 - c + \mathrm{i}\epsilon)^{\alpha} \tag{B3}$$

by differentiating with respect to α and setting $\alpha = 0$. The last integral is given by

$$L = \frac{\mu^{4-n}}{2(2\sqrt{\pi})^n}\,(-1)^{\alpha}\,c^{\frac{n}{2}+\alpha}\,\frac{\Gamma\left(-\alpha - \frac{n}{2}\right)}{\Gamma(-\alpha)}\,. \tag{B4}$$

Since $1/\Gamma(0) = 0$, the only non-vanishing contribution to K involves the differentiation of $\Gamma(-\alpha)$. Thus

$$K = -\frac{\mu^{4-n}}{2(2\sqrt{\pi})^n}\,c^{\frac{n}{2}}\,\Gamma\left(-\frac{n}{2}\right) \tag{B5}$$

where we have employed $\lim_{\alpha\to 0} \psi(-\alpha)/\Gamma(-\alpha) = -1$. We see that K contains poles at $n = 0, 2, 4$ which, according to the rules of correspondence, indicate quartic, quadratic and logarithmic ultraviolet singularities. However, (B1) involves the difference of two integrals of the K type and we find

$$V(\phi_c) = \tag{B6}$$
$$-\frac{\mu^{4-n}}{2(2\sqrt{\pi})^n}\left[\left(m^2 + \frac{\lambda\phi_c^2}{2}\right)^{n/2} - (m^2)^{n/2}\right]\Gamma\left(-\frac{n}{2}\right).$$

Clearly, the residue of the $n = 0$ pole cancels in (B6) and the leading singularity is given by the $n = 2$ pole:

$$V(\phi_c) = \frac{\mu^2}{8\pi}\frac{\lambda\,\phi_c^2}{(2-n)} + \cdots\,, \tag{B7}$$

which corresponds to a quadratic divergence. This conforms with the result for the leading singularity obtained by expanding (B1) in powers of $\lambda\phi_c^2$. Moreover, if one approaches the pole from below, as discussed after (34), the signs also coincide! As it is well known, the quadratic and logarithmic singularities in (B6) are cancelled by the δm^2 and $\delta\lambda$ counterterms [18].

The K-integral with $c = m^2$ is also interesting because in some formulations it is directly linked to the vacuum energy density contribution from free scalar fields [19]. In order to obtain a four-dimensional representation of (B2), we introduce a Feynman regulator $[\Lambda^2/(\Lambda^2 - k^2 - \mathrm{i}\epsilon)]^3$ and perform a Wick rotation, which leads to the Euclidean-space expression

$$K = \frac{1}{32\pi^2}\int_0^{\infty} \mathrm{d}u\; u \ln\left(\frac{u + m^2}{\sigma^2}\right)\left(\frac{\Lambda^2}{\Lambda^2 + u}\right)^3. \tag{B8}$$

In order to give mathematical meaning to the logarithm, in (B8) we have introduced a squared-mass scale σ^2 which, for the moment, is unspecified. Evaluating (B8), we have

$$K = \frac{1}{64\pi^2}\left\{\Lambda^4\left[\ln\left(\frac{\Lambda^2}{\sigma^2}\right) + 1\right] + m^2\Lambda^2 - m^4\left[\ln\left(\frac{\Lambda^2}{m^2}\right) - 1\right]\right\}, \tag{B9}$$

where we have neglected terms of $\mathcal{O}(m^2/\Lambda^2)$. In a free field theory calculation, one expects the answer to depend on

m^2 and Λ^2, as we found, for instance, in (21) and (27). By arguments analogous to those explained at the end of Sect. 3, one finds that σ^2 cannot be identified with m^2. In fact, with $c = m^2$, (B5) is proportional to $(m^2)^{n/2}$. Differentiating with respect to m^2, the $n = 0$ pole in (B5) cancels. This implies that terms of $\mathcal{O}(\Lambda^4 \ln(\Lambda^2/m^2))$ cannot be present, since otherwise contributions of $\mathcal{O}(\Lambda^4)$ would survive the differentiation with respect to m^2. An attractive idea to fix σ^2 is to invoke symmetry considerations. In particular, since according to the arguments of this paper the terms of $\mathcal{O}(\Lambda^4)$ violate the scale invariance of free field theories in the massless limit, we may choose $\sigma = \sqrt{e}\,\Lambda$ to eliminate such contributions. In that case, (B9) reduces to our previous result in (27), obtained by more elementary and transparent means! Correspondingly, in the DR version of K, the $n = 0$ pole $\mu^4(4-n)/4n$ may be removed in order to conform with the scale invariance of free field theories in the massless limit. This can be achieved by appending an $n/4$ normalization factor to the RHS of (B5), in which case the rules of correspondence between the DR and four-dimensional calculations reduce precisely to (34) and (34).

In summary, aside from the fact that the derivation of (12) and the calculation of (27) are particularly simple, they offer the additional advantage that they explicitly exhibit the partial scale invariance of free field theories.

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